G tree $\Rightarrow \gamma(G) \leq 4$
$\gamma(P_1)=1$, $\gamma(P_2)=\gamma(P_3)=2$, $\gamma(P_n)=3$ if $n \geq 4$
$\gamma(4\text{-pode}) = 4$

G cycle $\Rightarrow \gamma(G)=3$
$\gamma(K_{1,r}) = 2 = \gamma(K_{\ell,1})$
$\gamma(K_{\ell,r})=3$ if $\ell \geq 2$ & $r \geq 2$

Graph Coloring Games

$\gamma(W_4)=\gamma(W_6)=$ ~~$\gamma(\ldots)$~~ $= \gamma(W_n) \neq 4$, $n \geq 8$ $\quad \gamma(W_5)$ & $\gamma(W_7)$ unknown?

Let's play a game. We start with the graph featured in Figure 0 and three colors; red, blue, and green. This is a two player game. The goal of Player One is to color every vertex so that no two colors are adjacent to each other. Player Two is attempting to make it so that Player One cannot color every vertex. Below is an example of how the game could turn out.

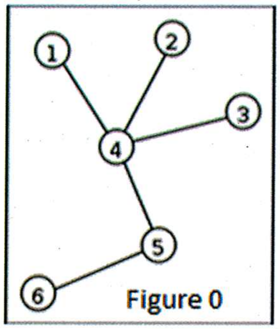


Figure 0

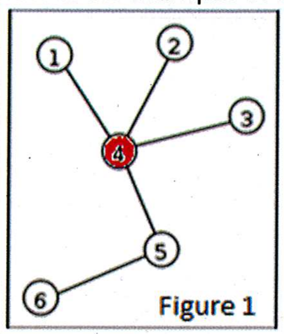


Figure 1

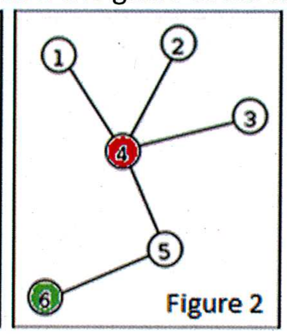


Figure 2

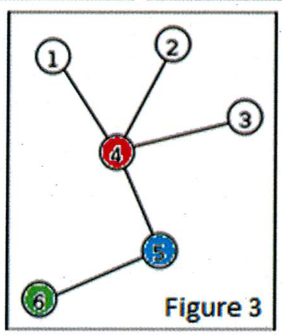


Figure 3

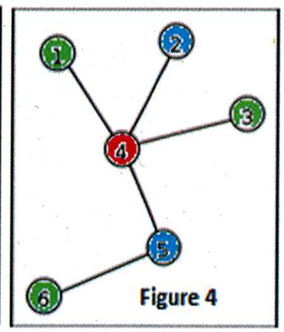


Figure 4

As shown above in Figure 1, Player One begins by coloring vertex 4 red. Player Two then colors vertex 6 with green. Player One's response is to color vertex 5 with blue. Vertices 1, 2, and 3 may be colored with either green or blue because they are adjacent to only the color red as seen in Figure 4.

Consider the following strategy for Player One:

- Initially color vertex 4 red.
  - If Player Two colors vertex 1, 2, or 3, color vertex 5 blue.
  - If Player Two colors vertex 5, color vertex 6 red.
  - If Player Two colors vertex 6, color vertex 5 with blue or green.
- After Player One's second move all remaining uncolored vertices are connected to at most one other vertex. Color accordingly; red, blue, green.

By coloring vertex 4, Player One has colored the only vertex that is connected to more than 2 other vertices. At most Player Two will be able to have two different colors connect to one uncolored vertex by coloring vertex 6. Hence, the above strategy ensures a win for Player One.

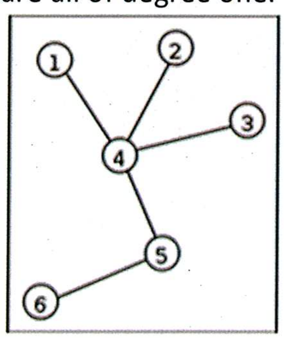
Let's try another game with the same graph but this time we're only going to use two colors, red and green. Player One initially colors vertex 4 red. Player Two then colors vertex 6 green as we see in Figure 2. However, at this point it is not possible for Player One to color vertex 5 green or red because it is adjacent to both available colors. Showing that Player Two has a winning strategy requires a more in-depth analysis of all the possible ways this game can be played. We need to consider every move Player One might make.

- Player One colors vertex 1, 2, or 3. Player Two's response is to color vertex 5 with the opposite color.
- Player One colors vertex 4. Player Two's response is to color vertex 6 with the opposite color.
- Player One colors vertex 5. Player Two's response is to color vertex 1,2, or 3 with the opposite color.
- Player One colors vertex 6. Player Two's response is to color vertex 4 with the opposite color.

Player Two's first move in all these situations presents Player One with an uncolored vertex connected to both available colors. At this point Player Two has accomplished his goal of preventing the other player of coloring the entire graph. Therefore the scenarios listed above prove that the described strategy is a winning strategy for Player Two because in every way the game can be played he has a response that leaves a vertex uncolorable.

Rules & Terminology

A *graph* consists of a set of vertices, V, and a set of edges, E. An *edge* is a pair of vertices. . Visually, we denote circles for vertices and lines for edges. Let G be the graph in Figure 0, therefore V= {1,2,3,4,5,6} and E = {{1,4},{2,4},{3,4},{4,5},{5,6}}. Two vertices connected by an edge are said to be *adjacent*. Let {a,b} be an edge. This means vertices a and b are adjacent. If a and b are adjacent vertices then the edge connecting them is said to be *incident* to a and b. A *subgraph* of a graph G is a graph whose vertex and edge set is a subset of that of G ~~that still maintains all G's properties~~. The degree of a vertex is the number of edges incident to that vertex. The degree of vertex 5 is 2. A vertex of degree 1 is defined as a leaf. In Figure 5 vertices 1,2,3,6 are leaves because they are all of degree one.

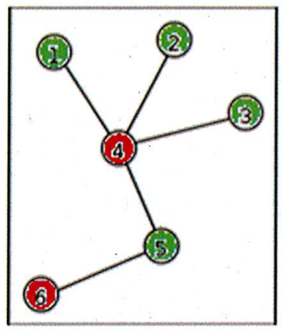


A *walk* from vertex a to b is an alternating list of vertices and edges in which each edge is incident with the vertices that come before and after it. A walk from Vertex 6 to 2 can be shown as w(~~6,2~~) = 6 , {6,5}, 5, {5,4}, 4, {4,2}, 2. We define the *length* of the walk, w, by the number of edges in the walk. The length of w = 3 because it crosses three edges; {5,6},{4,5}, and {2,4}. The walk shown from vertex 6 to 4 is also as a path. A *path* is a walk in which no vertex appears more than once. Paths can be written as an alternating list of vertices only. The path from vertex 6 to 2 is (6 -> 5 -> 4 -> 2)

Graph coloring games are comprised of a set of colors and a graph that two players compete to color. The goal, for Player One, is to color every vertex using the set of colors given so that no two adjacent vertices share the same color. In Figure 2 it would be an illegal move for Player One to color vertex 5 with either red or green. Player Two's goal is to color the graph in a way that there are uncolorable vertices. An *uncolorable* vertex is an open vertex that is adjacent to every color available to the players.

A *strategy* is a complete and unambiguous description of what to do in every possible situation. A *winning strategy* is a strategy that will guarantee a win for the player using it, regardless of what strategy the opposing player is using. In both games we played, each player has a winning strategy. We showed in the previous section that Player One has a winning strategy for the game on the graph in Figure 0, w/ 3 colors and P II ...

The *chromatic number* of a particular graph is the fewest number of colors necessary to be able to color every vertex with no adjacent vertices being the same color. For example, the chromatic number of the graph below is 2. This is more formally written as $\chi(G) = 2$, where G is the graph being analyzed. There is only one case where a graph would have a chromatic number less than 2. This is the where the graph contains only one vertex, therefore coloring the entire graph requires only one color.



Theorem 0: If G is a graph with at least two adjacent vertices, then $\chi(G) \geq 2$

Proof.

Suppose G is a graph with at least two adjacent vertices. It is impossible for both vertices to be colored with only one color. Therefore the chromatic number must be greater than or equal to 2.

Q.E.D.

The *game-chromatic number*, denoted by the Greek symbol γ, represents the least number of colors necessary for Player One to have a winning strategy in the corresponding graph-coloring game. The game-chromatic number of the graph in Figure 0 we have shown is 3. This is shown as $\gamma(G) = 3$.

Theorem 1: $\gamma(G) \geq \chi(G)$

Proof

If the game-chromatic number is less than the chromatic number then Player One has a winning strategy with a smaller set of colors than the graph is able to be colored with. This is a contradiction therefore the game-chromatic number must be greater than or equal to chromatic number.

Q.E.D.

Theorem 2: $\gamma(G) \leq N+1$, where N is the max degree of the graph.

Proof

Let G be a graph. Consider the graph-coloring game on G with $K \geq N + 1$ colors where N is the max degree of G. That means, at most it is possible for N different colors to be adjacent to one vertex. If Player One has at least N+1 different colors available for the game it is impossible to create an uncolorable vertex therefore the game-chromatic number will never be greater than $N + 1$.

Path Graphs

A *path* graph is a graph that can be drawn such that all vertices lie in a single straight line. A path graph, P_n, with n vertices has 2 leaves and n-2 vertices all of degree 2. For example, the figure below is a Path graph with 7 vertices; therefore there are 2 leaves and 5 vertices of degree 2. The leaves are marked blue, and the remaining degree 2 vertices are marked green.

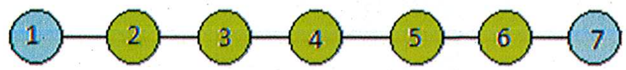


Figure Path

In an attempt to fully characterize path graphs we separate the class into three different categories.

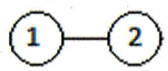
Theorem 3: $\gamma(P_1) = 1$, $\gamma(P_2) = \gamma(P_3) = 2$, and $\gamma(P_n) = 3$ if $n \geq 4$

Proof.

Suppose G is a path graph. Analyzing a graph-coloring game using the variable graph G, we will determine the different game-chromatic numbers of path graphs.

Figure 11

Let G = P_1 then G is a graph with only one vertex. Player One's winning strategy is to color vertex 1. Therefore the game-chromatic number is 1.



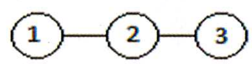


Figure 12 **Figure 13**

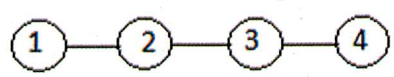
Let G = P_2 then G is a graph with two vertices. Player One needs to have at least two colors available to him/her to have a winning strategy.

- Color Vertex 1 with the first color, Player 2 must color vertex 2 color 2.

Let G = P_3. Player One needs to have at least two colors available to him/her to have a winning strategy.

- Color Vertex 2 with the first color. Vertex 3 and 1 may then both be colored by the second color.

Therefore it is sufficient for Player One to have a winning strategy if there are only 2 colors available in the corresponding graph-coloring game.



Let G = P_n and $n \geq 4$. Suppose $\gamma(P_n) = 2$. Player One will attempt the following strategy.

- Initially color vertex 1
  - Player Two colors vertex 3 the opposite color.
- Initially color vertex 2
  - Player Two colors vertex 4 the opposite color.
- Initially color vertex 3
  - Player Two colors vertex 1 the opposite color.

- Initially color vertex 4
    - Player Two colors vertex 2 the opposite color.

In each case, if only two colors are available to Player One he/she will lose therefore the game-chromatic number must be at least 3. Player One has a winning strategy with 3 colors because the max degree of G is 2. It is not possible for Player Two to create an uncolored vertex with three different colors adjacent to it.

Q.E.D

Theorem 4: If G is a path graph of at least 2 vertices,then $\chi(G) = 2$

Proof

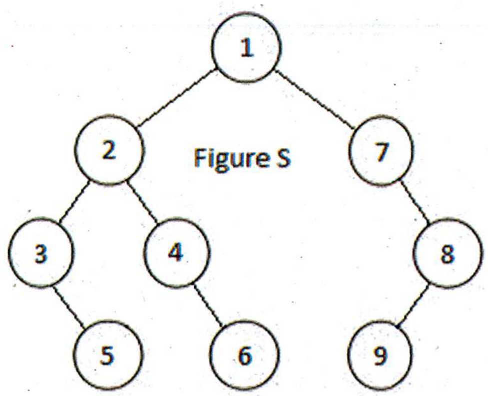
Let G be a path graph. Begin coloring one of the leaves and coloring it blue and coloring every uncolored vertex adjacent to it red. Then color every uncolored vertex adjacent the red vertices blue. Much like tree graphs, Path graphs have the characteristic of there existing a unique path between two vertices. Therefore by beginning coloring at the extreme side of a graph G, each newly colored vertex will adjacent to only one other colored vertex, the direction from which it came.

specialize since trees now come <u>after</u> paths.

Q.E.D.

Tree Graphs

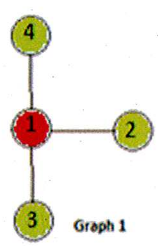
A *tree* is a graph that for every pair of vertices there exists a unique path between them. The graph below, Figure S, is a tree. This is because for each pair of vertices there is only one path between them. For example, between vertices 1 and 6 the path is 1,2,4,6.



Taking a systematic approach we discuss generally the chromatic number, game-chromatic number, and winning strategies for both Players under a variety of conditions.

Beginning with the most basic trees is the first step in analyzing the class as a whole. The following are analyses of two basic trees and their properties.

Graph 1:



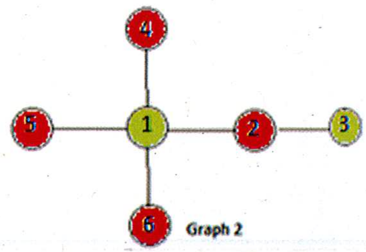
χ(Graph 1) = 2. Visually this is shown above. By coloring the center vertex red, each remaining open vertex must be colored green.

γ (Graph 1) = 2. In order to prove this we must show that Player One has a winning strategy in a graph-coloring game using graph one and at least two colors.

- Initially color the center vertex Red
- Color any remaining vertices green.

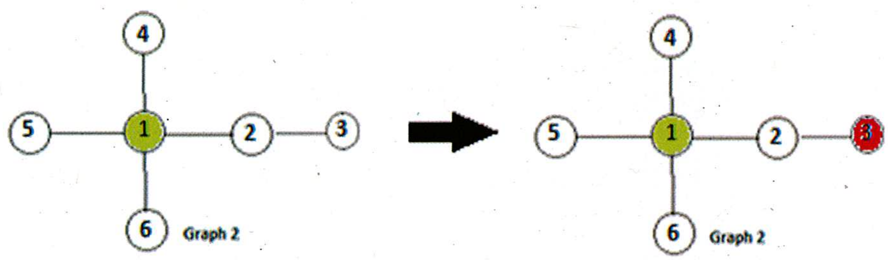
If Player One initially colors the central vertex red then the remaining vertices are leaves, who are only adjacent to one vertex, the one that is currently red. Therefore they cannot be colored red and in order to complete coloring the graph then 2 colors is sufficient for Player One to win. Player Two is limited to coloring the remaining leaves green.

Graph 2:



χ (Graph 2) = 2. Visually this is shown above. Beginning at the right-most vertex and coloring it red then alternating between green and red for subsequent adjacent vertices.

γ(Graph 2) = 3. The Gamma for Graph 2 must be greater than two. To prove this we assume that in a graph-coloring game with the corresponding graph and two colors that Player One has a winning strategy. However, as seen in the example below, regardless of which vertex is colored by Player One, Player Two has the ability to color a vertex with the opposite color such that an open vertex is adjacent to both colors.



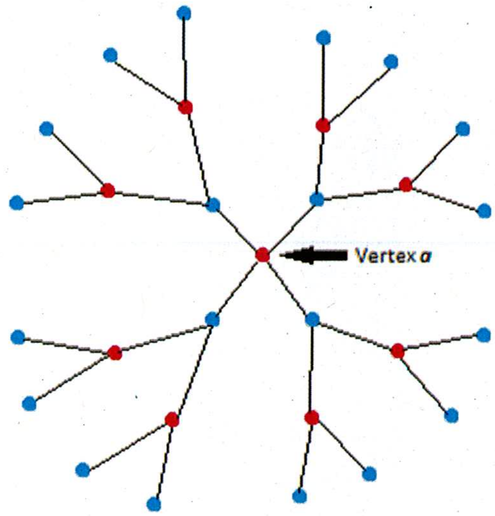
So we can safely assume that Player Two has a winning strategy in the corresponding graph with 2 colors. In order to analyze the aspects of this graph more carefully we repeat the graph-coloring game with the same graph but instead using three colors, red, green, and blue. Player One has a winning strategy:

- Color Vertex 1 green
- Color remaining vertices with available colors.

This is a winning strategy for Player One because once vertex 1 has been colored there are no uncolored vertices which are of degree three. Therefore Player Two will never have the opportunity to create an uncolored vertex adjacent to three different colors.

In the examples of Graph 1 and 2 we saw no variation in the chromatic number of these graphs. This has to do primarily with the definition of trees themselves. The only case where the chromatic number of a tree is not 2, is when the graph is a single vertex.

Theorem 5: If G is a tree with more than one vertex, then $\chi(G) = 2$.



Proof

Let G be a tree. The figure above is an example of such a graph. We begin by coloring an arbitrary vertex, vertex *a,* red. Then color each vertex adjacent to *a* the second color, blue. Next we color all the vertices adjacent to blue vertices with red. This continues until all vertices are colored or until we are unable to color with only the two colors.

Let us assume instead that we are unable to color every vertex in G with only 2 colors. This means that at some point, coloring adjacent vertices, there was an open vertex adjacent to both colors. This means 2 of the paths that began with vertex *a* have met at an open vertex on the graph. Then in this graph there is more than one between any two given vertices. This shows that G cannot be a tree and therefore a contradiction.

By the definition of trees we know there exists a unique path between any vertices in G. Therefore there exists a unique path from vertex *a* to every other vertex on the graph. This means in addition to two colored paths never becoming adjacent to the same open vertex on the graph, every other vertex has a path to *a* showing every vertex will at some point be colored.

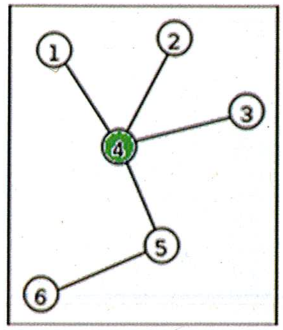
Q.E.D

A partially colored tree graph, G, can be divided into what we call components. Given an uncolored vertex v, the component that contains V is the maximal subtree for which colored vertices are leaves. A *component* is a maximal subtree of G that shares one colored vertex and no uncolored vertices with no its leaves adjacent subtrees. A *subtree* is a subgraph of a tree. Below is a visual example that further elucidates what components are.

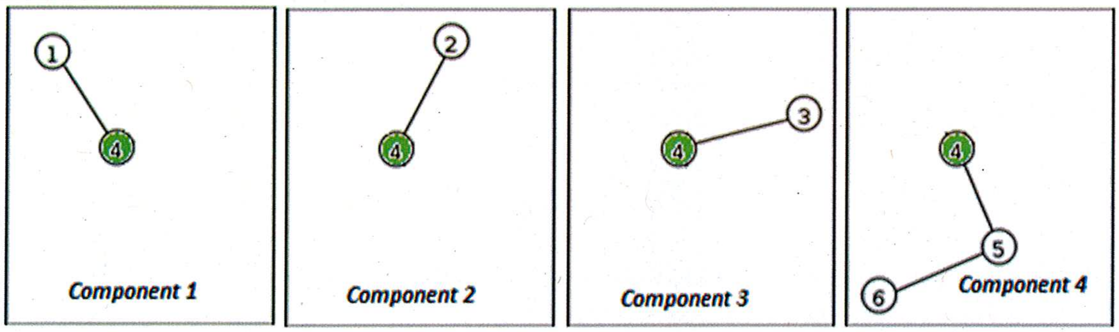
that is itself a tree.

Place here or ~~after~~ just before Thm 6.

Since the components partition the set of uncolored vertices into disjoint sets, during ~~the~~ his turn a player will color a vertex in ~~a~~ exactly one component.



Coloring vertex 4 splits the graph into four unique components that have only one common vertex



We now move into a more general analysis of trees in graph-coloring games and apply their results in ways to try and fully characterize the clas s.

Theorem 6: If G is a tree, then $\gamma(G) \leq 4$

Proof.

Suppose G is a tree. Consider the graph-coloring game using G and a set of colors. We will show that Player One can always color a vertex so that no components contain more than 2 colored vertices.

Player One's strategy is:

- If there are no components with one colored vertex, color anywhere. Now all components will have one colored vertex.
- If there are only components with one or two colored vertices, color adjacent to a colored vertex in ~~a one colored vertex~~ component.
- If there is a component with and three colored vertices, color so that no component contains more than 2 colored vertices.

We will now go through a graph-coloring game step-by-step and apply Player One's strategy and see the result.

1. Player One colors a vertex and now all components have one color
2. Player Two colors a vertex and creates one component with two colors. ed vertices.
   - Player One colors adjacent to a colored vertex in a one color component, unless there are no one color components; For example this would be the case later in the game where Player Two has made every component have two colored

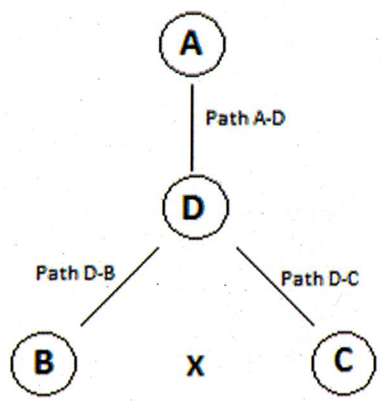
This is more specific than the stated strategy.

vertices. If so Player One should color adjacent to a colored vertex in a two color component.

3. Player Two colors a vertex and the graph has several components with one or two colored vertices and at most one component with three colored vertices.

   In order to show Player One has an available move to him we must consider the following lemma and apply its results to the theorem we are going to prove.
   Lemma 1: If X is a component of a tree containing three colored vertices, A, B, and C, then there exists a unique vertex ***z*** such that z can be colored so that each resulting component contains no more 2 colored vertices.

4.



5.

Suppose G is a tree, X is a component of G, and A,B,C are three colored vertices in X. Since colored vertices are leaves of a component, we know every path in X starting from A begins with at least the first two vertices being identical. On a path from A to B and A to C there exists a vertex P which is the last shared vertex between these two paths.

The paths from P to B and P to C have only one common vertex, P. Indeed, suppose that instead that the paths P to B and P to C shared a second common vertex Q. Then B to P to C and B to Q to Q would be two paths from B to C, a contradiction to the graph being a tree.

Therefore we know that on the paths A to B, A to C, and B to C there is only one common vertex, P. In coloring P the graph is split into 3 or more components containing no more than 2 colored vertices.

Q.E.D.

- Player One colors a vertex in the component with three colored vertices such that the resulting components contain no more than two colored vertices. There exists a unique vertex, as shown in Lemma 1, such that Player One can always turn a three color component into several two and one color components.

This is a full strategy for Player One because in every possible case he/she has a response to a move done by Player Two. Also as shown in Lemma 1, there is always an available vertex in which Player Step 3 will be repeated until at some point there are only components with two colored vertices and no one colored components. However, as seen in Lemma 1 there always exists a vertex in a three colored component such that the resulting graph has components with 2 colored vertices or less. If the most colored vertices in a component at any

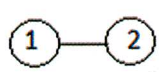
given time is 3, then the most different colors ~~in~~ possible in a component is 3. Therefore it is possible that Player One would need to use a fourth color to return the graph to components containing only three colors. So γ(G) ≤ 4.
Q.E.D.

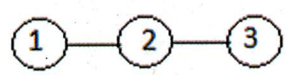
The theorem above can potentially help create a full characterization for tree graphs. This would allow us to determine which trees correspond to differing game-chromatic numbers. One would be able to quickly analyze a tree and state assuredly for what numbers of colors Player One has a winning strategy.
There are four different ways trees can be categorized by utilizing a graph-coloring game using G, a tree. The four different categories are γ(G)=1, γ(G)=2, γ(G)=3, γ(G) = 4.

γ(G) = 1

Let G be a tree. In order for the game-chromatic number to be equal to one, the chromatic number must also be equal to one. The only case where this happens is if G only has one vertex.

γ(G)=2

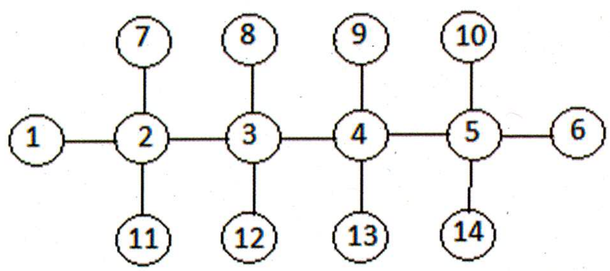




Let G be a tree. In order for the game-chromatic number to be equal to two then G must contain P_2 or P_3 . Player One must have at least 2 different colors available in order have a winning strategy.

γ(G) = 3
Let G be a tree. In order for the game-chromatic number to be equal to 3, G must contain at least a P_4 but not the figure below.

γ(G) = 4



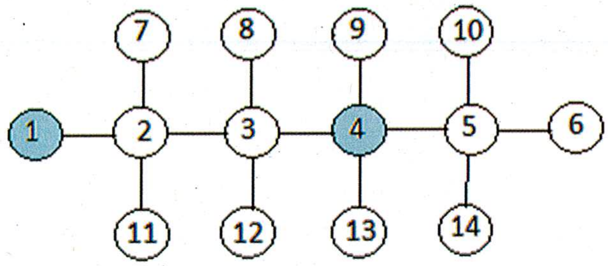
Consider the diagram above. The game-chromatic number of the graph above is 4. This can be shown by playing a graph-coloring game using the graph above and 3 colors.

Player One's strategy is to initially color one of the central vertices, 1,2...,6. Need to also consider a leaf.
Player Two responds by coloring a vertex, 2 vertices away.

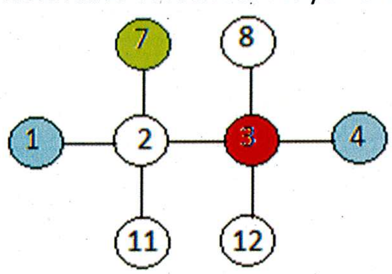
- If Player One colors vertex 1, Player Two colors vertex 4 the same color.
- If Player One colors vertex 2, Player Two colors vertex 5 the same color.

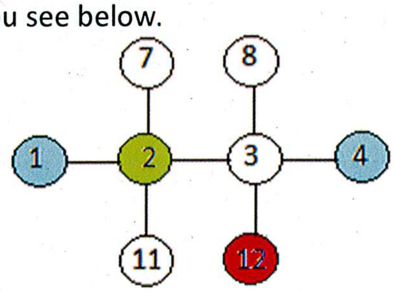
- If Player One colors vertex 3, Player Two colors vertex 6 the same color.
- If Player One colors vertex 4, Player Two colors vertex 1 the same color.
- If Player One colors vertex 5, Player Two colors vertex 2 the same color.
- If Player One colors vertex 6, Player Two colors vertex 3 the same color.

Now, we can generalize this strategy in a way that will simplify the results. Player Two is always able to create a component that looks like the figure below.

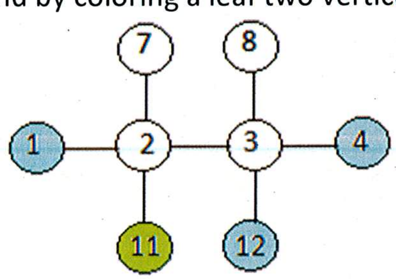


In doing so, Player One is presented with several options. Player One can color outside the component thus leaving what is inside untouched. Player Two would then choose to color vertex 7 green. This forces Player One to color vertex 2 in order to prevent an uncolorable vertex from being created but assists in the process nonetheless. In coloring vertex 2, vertex 3 is now adjacent to two different colors and two uncolored vertices which Player Two can color and make vertex 3 uncolorable.
The more important case is where Player One chooses to color inside the component:
Player One should not color vertices 2 or 3 using a second color because Player Two would respond by coloring one of the leaves of the adjacent uncolored vertices using a third color thus creating an uncolorable vertex for Player One as you see below.

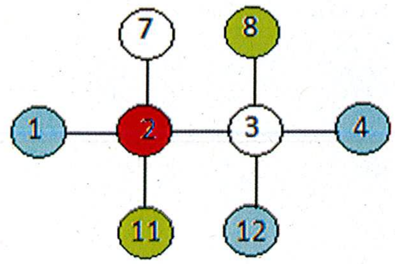




Thus one option for Player One would be to color one of the leaves of the component; vertices 7, 8, 11, or 12. Using a second color would only help Player Two so the choice is to color one of these vertices using the same color already present in the component. Since this component is symmetrical we need not look at each individual case for coloring. If Player One colors one of the leaves Player Two will respond by coloring a leaf two vertices away with a second color.



Player One again always has the option to ignore this component and color outside however it would only benefit Player Two. Player One must color vertex 2. If he does not, Player Two will be able to color vertices 3 or 7 using a third color and create an uncolorable vertex. This presents a dilemma though, in coloring vertex 2 Player One must use his third color, which then vertex 3 is now adjacent to two different colors which, as seen in the case above would allow Player Two to color vertex 8 and make vertex 3 uncolorable.

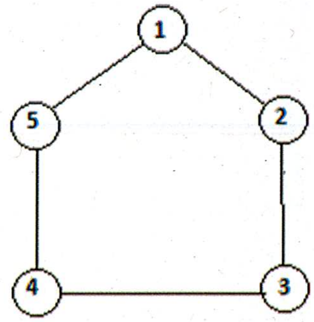


We have therefore arrived a point where Player One cannot win with three colors.
While the first three characterizations are true, and can be shown, the last merely implies that if our initial graph is contained in a larger graph G then the game-chromatic number will be four. What has yet to be found is that if this graph is not a subgraph of G then the game-chromatic number must be 3 or less.

If the prime graph is a proper subgraph of G, might ~~[illegible]~~ PI play so that PII is the first to color in the pede. Will PII still have a winning strategy in that circumstance?

Cycle Graphs

A *cycle* graph is a graph that contains three or more vertices connected in a single closed loop. Below is an example of a cycle graph containing a loop of 5 vertices each of degree 2. It is not possible for a cycle to have less than 3 vertices. If there are less than 3 vertices, there cannot be a loop and every vertex will not be degree 2.



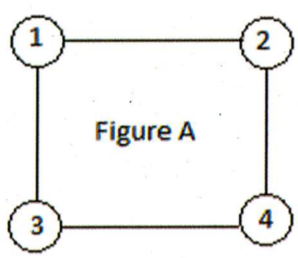
Theorem 7: If G is a cycle, then the degree of every vertex is 2.

Proof

Let G be a cycle graph and assume that there exists a vertex *a* which is at least degree 3. This means the third vertex connected to *a* is either part of an interconnected loop or it is a leaf. Both of these conditions would violate the definition of cycles. There no longer would be a single closed loop. Therefore if G is a cycle then the degree of every vertex is 2.

Q.E.D.

There are two important classes of cycle graphs; cycles containing an even number of vertices, and cycles containing an odd number of vertices . This accounts for every possible variation of cycle graph.



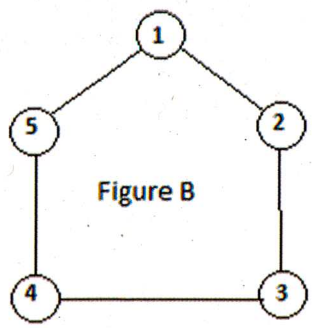
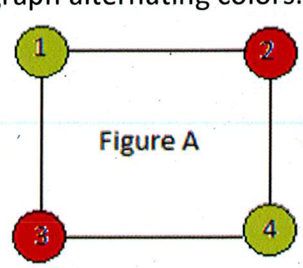
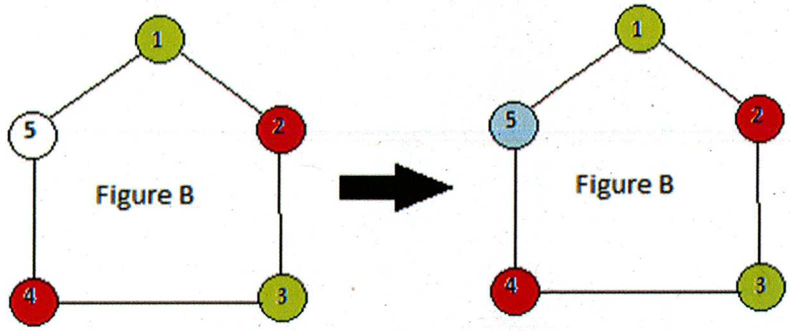


Figure A and Figure B are representative of all classes of cycle graphs. In attempting to determine the chromatic number of Figure A we begin by coloring an initial point and from there alternating between the two available colors. Begin by coloring vertex 1 green and move clockwise around the cycle graph alternating colors. The result is:



The chromatic number of Figure A is 2. As shown in Theorem 0.1 it is not possible for the chromatic number to be equal to 1 in a graph with at least two adjacent vertices.

In attempting the same method with Figure B we reach a different result. Begin by coloring Vertex 1 and color clockwise, alternating colors every vertex. The result is:



In Figure B it is clear that using 2 colors is not adequate for the graph to be fully colored. Vertex 5, when using this method and only 2 colors is uncolorable. Therefore the chromatic number of Figure B must be 3. Generalizing the two classes of cycles we can show for all cases what the chromatic number of varying cycles will be.

Theorem 8: If G is a cycle with an even number of vertices, $\chi(G) = 2$

Proof.

Suppose G is a cycle with an even number, A, of vertices. Let vertex 1 be the starting point in the graph for coloring. Clockwise around the cycle from vertex 1 each vertex is labeled sequentially up to A.

Color Vertex 1 Red. Alternate red and green around the cycle until reaching vertex 1. As defined, all vertices that are odd will be red, and all vertices that are even will be green. The only case where two even numbers, or two odd numbers would be adjacent is if 1 and A are the same.

As defined earlier, A is even therefore if G is a cycle with an even number of vertices $\chi(G) = 2$.

Q.E.D.

Theorem 9: If G is a cycle with an odd number of vertices, $\chi(G) = 3$

Proof.

Suppose G is a cycle with an odd number, B, of vertices. Let vertex 1 be the starting point in the graph for coloring. Clockwise around the cycle from vertex 1 each vertex is labeled sequentially up to B.

Color Vertex 1 Red. Alternate red and green around the cycle until reaching vertex 1. As defined, all vertices that are odd will be red, and all vertices that are even will be green. The only case where two even numbers or two odd numbers would be adjacent is if 1 and B are the same.

As defined earlier, B is odd, therefore if only 2 colors are available and coloring begins at vertex 1 then vertex B is uncolorable. In order for vertex B to be colored then we must use a third color. If G is a cycle with an odd number of vertices then $\chi(G) = 3$.

Q.E.D.

Theorem 10: If G is a cycle , then $\gamma(G) = 3$

Let G be a cycle. Let us analyze a graph-coloring game using 2 colors, red and blue. We assume Player Two has a winning strategy.

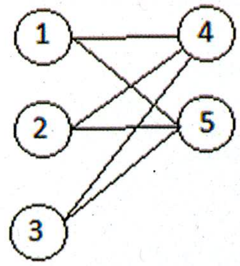
- Initially Player One colors a vertex m, red.
  - Adjacent to m there exists two open vertices, as shown by Theorem 8, which are now adjacent to the color red.
- Player Two colors adjacent to one of the open vertices adjacent to vertex m with blue.

According to the Theorem 8, every vertex in G is of degree 2 therefore there exists a vertex such that if Player Two colors it with the second color an uncolorable vertex is created.

Therefore γ(G) must be greater than 2. We will now repeat the same graph-coloring game with 3 colors.. The max degree of G is 2. As shown in Theorem 2, the game-chromatic number ≤ max degree of G plus one which, in this case, implies the game-chromatic number is ≤ 3. We have already shown the game-chromatic number is not 2, hence, it must be 3.
Q.E.D.

Complete Bipartite Graphs

A *complete bipartite* graph is a graph with two non-empty sets of vertices, L and R, such that every pair of vertices, one from L and one from R, are adjacent. Complete bipartite graphs are denoted as $K_{l,r}$ where l and r are the sizes of the two sets of vertices.



The graph above is the complete bipartite graph, $K_{3,2}$, where$|L| = 3$ and $|R| = 2$. The set of vertices, L, are placed on the left while the the set, R, are on the right. The figure above clearly illustrates the adjacency relationship between both L and R.

Theorem 11: If G is a complete bipartite graph then χ(G) = 2.
Proof.

Suppose G is a complete bipartite graph, $K_{l,r}$. By the definition of complete bipartite graphs we know that all vertices in L are adjacent to all vertices in R. Therefore since there is no inter-adjacency within L, all vertices in L can be colored using one color. Logically then, all vertices in R can be colored using a second color. Therefore, the chromatic number of all complete bipartite graphs is 2.

Theorem 12: If G is a complete bipartite graph where L or R = 1 then γ(G) =2.
Proof.

Let G be a complete bipartite graph, $K_{l,r}$, where $|L| = 1$ and $|R| \geq 1$. Consider the game using G and 2 colors. Player One colors the vertex in the set of vertices L n. Therefore, the entire set of R q vertices can be colored using the second color no matter what Player Two's response is.

Let G be a complete bipartite graph, $K_{n,q}$, where $|L| \geq 1$ and $|R| = 1$. Consider the game using G and 2 colors. Using the same argument, Player One initially colors the only vertex in the set of q vertices. This allows for the entire set of n vertices to be colored by any color. Therefore in both cases, the game-chromatic number of complete bipartite graphs when $n \text{ or } q = 1$ is 2.

Theorem 13: If G is a complete bipartite graph where l and r ≥ 2, then γ(G) = 3.
Proof.

Let G be a complete bipartite graph, $K_{l,r}$, where $n \geq 2$ and $q \geq 2$. Consider the game using G and only 2 colors, red and green.

- Player One colors a vertex in$L$ green.
- Player Two responds by coloring a second vertex in L red

All the vertices in R are now uncolorable. Player One needs at least 3 colors in this situation.

Therefore, now take a look at the game using G but this time with 3 colors, red, green, and blue:

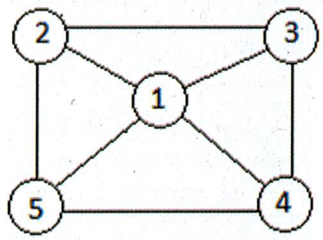
- Initially Player One colors a vertex in L green
- Player Two responds by coloring a second vertex in L red. ← Need to consider all possible moves by P II.
- Player One's second move is to color a vertex in$R$ blue.

From this point, the entire set of vertices R q can be colored blue and the entire set, L n, can be colored either green or red. It is not possible for Player Two to force a fourth color no matter how large L or$R$ are. Therefore for the graph-coloring game using, $K_{l,r}$, where $n \geq 2$ and $q \geq 2$ the game-chromatic number is 3.

Q.E.D

Wheel Graphs

A *wheel graph*, more commonly referred to as an n-wheel, is a graph that contains a cycle of n-1 vertices all connected to a single vertex. Therefore degree of the central vertex is n – 1, and all remaining vertices are of degree 3.



Represented in the figure above is a 5-wheel. There is a cycle of 4 vertices all connected to vertex 1. Vertices 2,3,4,5 as described above then are degree 3 whereas vertex 1 is degree 4. These are the basic properties that are required for a graph to be a wheel.

Wheels can be separated into two different categories; K-Wheels in which the k-1 cycle is even and the other is where the k-1 cycle is odd. This the distinguishing characteristic between wheels.

Theorem 14: If G is K-Wheel where K is even number greater than or equal to 4, then χ(G) = 4 and γ(G) = 4.
Proof.

Suppose G is a K-wheel containing a K-1 cycle, where K-1 is odd. Coloring an odd numbered cycle requires 3 colors. This is shown in Theorem 6. Hence to color the center vertex, which is adjacent to the entire cycle, we need a fourth color.

In order to determine the game-chromatic number for a graph-coloring game using G we reference back to our initial definitions. The game-chromatic number must be at least equal to the chromatic number of the graph.

In a graph-coloring game using G and 4 colors; red, blue, green, and orange, we begin with the following Strategy for Player One:

- Initially color the central vertex red.
- Color remaining vertices as allowed.

Following Player One's initial move all remaining uncolored vertices are now adjacent to only two uncolored vertices. All outer vertices cannot be colored red which leaves only 3 available colors. Therefore it is impossible for Player Two to color such that there is an uncolored vertex adjacent to 4 different colors.

Q.E.D.

Theorem 15: If G is a K-Wheel where K > 7 and is ~~always~~ odd, then χ(G) = 3 and γ(G) = 4.
Proof.

Let G be a K-wheel which contains a K-1 cycle, with K-1 ~~always~~ being even. As shown in Theorem 5, coloring an even numbered cycle requires 2 colors. Hence to color the final vertex, which is adjacent to the entire cycle, a third color is needed. So, the chromatic number of G is 3.

In order to determine the game-chromatic number for a graph-coloring game using G we reference back to our initial theorems. The game-chromatic number must be at least equal to the chromatic number of the graph.

P1

In a graph-coloring game using G and 3 colors; red, blue, green, we begin with Player One.

- Initially color the central vertex red.
- Player Two colors an arbitrary vertex, Z, green.

There are now two uncolored vertices adjacent to two different colors. Player One can choose to color adjacent to vertex Z using a third color. However, since K ≥ 6, Player Two can color adjacent to the remaining uncolored vertex with ~~the~~ orange thereby making it uncolorable. If Player One chooses not to color adjacent to vertex Z then by the same argument Player Two can force Player One to use a fourth color.

There is one alternate case that must be analyzed as well.

- Player One colors an arbitrary vertex, V, red.
- Player Two then colors the central vertex orange.

There exists two uncolored vertices, adjacent to V, that are adjacent to two different colors. It is not possible for Player One to color both vertices adjacent to V therefore Player Two on his/her following turn can color one of the uncolored vertices adjacent to V with orange. This means for Player One to be able to win he/she must have at least 4 different colors available.

Hence, as shown in Theorem S, in graph-coloring game using G and 4 colors Player One has a winning strategy. By initially coloring the central vertex Player One has colored the only vertex with a degree greater than 4. Therefore it is not possible for Player Two to create an uncolorable vertex. So, the game-chromatic number of K-Wheels, where K is odd, is 4.

Q.E.D